Detector Completeness for Finite Blaschke Packets via Gram–Matrix Nondegeneracy

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07 May 2026

Abstract

We construct an explicit finite-rank reproducing-kernel detector for the model space $K_{B_P} = H^2/B_P H^2$ associated with a finite Blaschke packet $P = \{w_1, \ldots, w_n\} \subset \mathbb{D}$. The principal observation is that the Gram matrix $G_{ij} = \langle k_{w_i}, k_{w_j} \rangle_{H^2}$ of the reproducing kernels at distinct packet points is a Cauchy-type matrix and is therefore nonsingular: in fact $\det G$ is, up to a positive multiplicative factor, the squared modulus of a generic Cauchy determinant $\prod_{i<j} |w_i - w_j|^2 / \prod_{i,j}(1 - w_i \overline{w_j})$. Nondegeneracy of G yields a rank-n family of evaluation functionals that separates every nonzero vector of K_{B_P}. We lift this finite-rank detector to a candidate inhabitant of the `RKHSModelSpaceDetector` interface used by the Burnol/Blaschke no-phantom programme, and isolate the residual completion obstruction in a paired `Vol6.Obstruction.FiniteRankDetectorObstruction` module. The argument is purely finite Hardy-space theory: it neither invokes Nyman density nor any equivalent of the Riemann hypothesis.

Contents

1 Introduction

1.1 Setting and motivation

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc and $H^2 = H^2(\mathbb{D})$ the classical Hardy space of holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ whose boundary values lie in $L^2(\mathbb{T})$. For each $w \in \mathbb{D}$ the function

$$k_w(z) \;:=\; \frac{1}{1 - \overline{w}z}$$

is the *reproducing kernel* at w: for every $f \in H^2$ one has the fundamental identity

$$\langle f,\, k_w\rangle_{H^2} \;=\; f(w).$$

A finite Blaschke product attached to a packet $P = \{w_1, \ldots, w_n\} \subset \mathbb{D}$ of distinct points is the function

$$B_P(z) \;=\; \prod_{i=1}^{n} \frac{w_i - z}{1 - \overline{w_i}z} \cdot \eta_i,$$

where each η_i is a unimodular phase (we shall always take $\eta_i = \overline{w_i}/|w_i|$ when $w_i \neq 0$ and $\eta_i = -1$ when $w_i = 0$). The choice of η_i affects only the specific holomorphic representative $B_P(z)$; the submodule $B_P H^2$ and hence the model space K_{B_P} are independent of the phases. The associated *model space* is the orthogonal complement

$$K_{B_P} \;=\; H^2 \;\ominus\; B_P H^2.$$

For finite packets it is a classical fact that $\dim K_{B_P} = n = \#P$ and that K_{B_P} is spanned by the kernels $\{k_{w_1}, \ldots, k_{w_n}\}$.

The present paper carries out the second proof-sprint step in the volume VI programme ([2]). The first paper constructed `FiniteBlaschkePacket` and the explicit model space K_{B_P}, and proved $K_{B_P} \neq 0$ for every nonempty packet by exhibiting a single reproducing kernel vector. The second step — the present paper — shows that the *full* family of reproducing kernels detects every nonzero vector of K_{B_P}, and that the corresponding Gram matrix is nondegenerate. This yields a finite-rank inhabitant of the `FiniteModelSpaceDetector` structure of [1, §2.7] and is the elementary cornerstone of the detector half of the no-phantom calculus.

1.2 Statement of results

Throughout, $P = \{w_1, \ldots, w_n\} \subset \mathbb{D}$ is a finite collection of distinct points and $K_{B_P} = H^2 \ominus B_P H^2$ is the associated finite-dimensional model space.

Theorem 1.1 (Gram-matrix nondegeneracy)**.** *The Hardy Gram matrix*

$$G_{ij} \;=\; \langle k_{w_i},\, k_{w_j}\rangle_{H^2} \;=\; \frac{1}{1-\overline{w_i}w_j}$$

of the reproducing kernels at the (distinct) packet points is Hermitian positive definite. In particular $\det G \neq 0$*.*

Theorem 1.2 (Detector completeness)**.** *For every nonzero $v \in K_{B_P}$ there exists $\alpha \in P$ such that $\langle v,\, k_\alpha\rangle_{H^2} \neq 0$. Equivalently, the family of evaluation functionals*

$$ev_\alpha : K_{B_P} \to \mathbb{C}, \qquad ev_\alpha(v) = \langle v,\, k_\alpha\rangle_{H^2} \;=\; v(\alpha),$$

is total: $\bigcap_{\alpha\in P} \ker ev_\alpha = \{0\}$*.*

Theorem 1.3 (Lift to a finite-rank detector)**.** *For every finite Blaschke packet P with nonempty* ZeroIndex*, the data (P, k, ev) define a term-mode inhabitant of* ***`FiniteModelSpaceDetector`*** *for the finite-packet defect object K_{B_P}. Concretely the rank equals $\#P$ and* detectsVector $v\,\alpha := ev_\alpha(v) \neq 0$*.*

The lift to the canonical Burnol/Blaschke `RKHSModelSpaceDetector` (operating on the opaque `burnolBlaschkeFactor.modelSpaceCarrier`) requires a colimit/completion step that is not available in vol5 (cf. [2, RF-05]). We therefore isolate the completion obstruction in `Vol6.Obstruction.FiniteRankDetectorObstruction` and discharge only the finite-packet specialisation here.

1.3 Outline

Section 2 reviews reproducing kernel Hilbert space (RKHS) theory in H^2. Section 3 recalls the finite Blaschke packet API of [3] and computes K_{B_P} explicitly. Section 4 proves the Cauchy-determinant formula for the Hardy Gram matrix, including positive definiteness and the explicit determinant (2). Section 5 assembles the finite-rank detector and proves Theorem 1.2. Section 6 states the residual completion obstruction. Section 7 describes the Lean formalisation. Section 8 reports the Haskell verification. Section 9 concludes.

2 Reproducing kernel structure of H^2

2.1 The Hardy space H^2

We adopt the following standard normalisation.

Definition 2.1. $H^2 = H^2(\mathbb{D})$ is the space of holomorphic functions $f(z) = \sum_{n\ge 0} a_n z^n$ on $\mathbb{D}$ such that

$$\|f\|^2_{H^2} \;:=\; \sum_{n=0}^{\infty} |a_n|^2 \;<\; \infty.$$

The inner product is $\langle f,\, g\rangle_{H^2} = \sum_{n\ge 0} a_n \overline{b_n}$ where $g(z) = \sum_{n\ge 0} b_n z^n$.

Lemma 2.2 (Reproducing property)**.** *For each* $w \in \mathbb{D}$ *define* $k_w(z) = 1/(1-\overline{w}z) = \sum_{n\geq 0} \overline{w}^n z^n$. *Then* $k_w \in H^2$, $\|k_w\|_{H^2}^2 = 1/(1-|w|^2)$, *and for every* $f \in H^2$,

$$\langle f,\, k_w\rangle_{H^2} \;=\; f(w).$$

Proof. Writing $f(z) = \sum a_n z^n$, $\langle f,\, k_w\rangle = \sum a_n \overline{\overline{w^n}} = \sum a_n w^n = f(w)$, and $\|k_w\|^2 = \sum |w|^{2n} = 1/(1-|w|^2)$. $\square$

Remark 2.3. The kernel k_w is sometimes called the *Szegő kernel*. It is holomorphic in $z \in \mathbb{D}$ and antiholomorphic in $w \in \mathbb{D}$. The map $w \mapsto k_w \in H^2$ is the (anti-linear) Riesz representer of the evaluation functional $f \mapsto f(w)$.

2.2 Inner products of two kernels

Lemma 2.4 (Two-point inner product)**.** *For* $u, w \in \mathbb{D}$,

$$\langle k_u,\, k_w\rangle_{H^2} \;=\; k_u(w) \;=\; \frac{1}{1-\overline{u}w}.$$

In particular, $\langle k_w,\, k_w\rangle = 1/(1-|w|^2) > 0$.

Proof. Apply the reproducing property of Theorem 2.2 to $f = k_u$ at the point w: $\langle k_u,\, k_w\rangle_{H^2} = k_u(w) = 1/(1-\overline{u}w)$. $\square$

Remark 2.5. We adopt the convention that the Hardy inner product is *conjugate-linear in the second argument* (cf. Theorem 2.1). With this convention, Theorem 2.4 reads $\langle k_u,\, k_w\rangle_{H^2} = 1/(1-\overline{u}w)$. We define the Gram matrix of the family $\{k_{w_i}\}_{i=1}^n$ by

$$G_{ij} \;:=\; \langle k_{w_i},\, k_{w_j}\rangle_{H^2} \;=\; \frac{1}{1-\overline{w_i}w_j} \;=\; \frac{1}{1-w_j\overline{w_i}}. \tag{1}$$

The two written forms of the denominator agree because complex multiplication is commutative; we shall freely use either form. Note that (1) is the standard Gram-matrix indexing $G_{ij} = \langle v_i,\, v_j\rangle$.

The matrix $G \in \mathbb{C}^{n\times n}$ defined by (1) is the *Hardy Gram matrix* of the family $\{k_{w_i}\}_{i=1}^n$ at the points $\{w_i\} \subset \mathbb{D}$. It is the central object of the present paper.

3 Finite Blaschke packets and their model spaces

3.1 Definitions

Definition 3.1 (Finite Blaschke packet)**.** A *finite Blaschke packet* is a tuple $P = (I, \mathrm{fin}, w, \theta)$ consisting of:

1. an index type $I = \mathrm{ZeroIndex}$ with finite cardinality $\#I = n$;
2. a finiteness witness $\mathrm{fin} : \mathrm{Fintype}\, I$;
3. an injection $w : I \to \mathbb{D}$, $i \mapsto w_i$;

4. a predicate $\theta : I \to \mathrm{Prop}$ flagging *off-critical* members.

The off-critical predicate θ plays no role in this paper; it is supplied for compatibility with the off-critical zero defect kernel construction ([2, Paper 03]).

Definition 3.2 (Blaschke product)**.** For a finite packet $P = \{w_1, \ldots, w_n\}$ the associated finite Blaschke product is

$$B_P(z) \;:=\; \prod_{i=1}^{n} b_{w_i}(z), \qquad b_w(z) \;:=\; \frac{w-z}{1-\overline{w}z} \;\cdot\; \eta(w),$$

with the unimodular phase $\eta(w)$ chosen so that $b_w(0) > 0$ when $w \neq 0$ and $b_0(z) = z$. We have $|B_P(\zeta)| = 1$ for $\zeta \in \mathbb{T}$, and the zero set of B_P in $\mathbb{D}$ is exactly P (counted with multiplicity, but here all w_i are distinct).

Definition 3.3 (Model space)**.** $K_{B_P} \;:=\; H^2 \,\ominus\, B_P H^2 \;=\; (B_P H^2)^{\perp} \subset H^2$.

3.2 Explicit basis

The following result is classical, going back to Takenaka and Malmquist; we record the proof for completeness.

Proposition 3.4 (Kernel basis of K_{B_P})**.** *K_{B_P} is an n-dimensional subspace of H^2 and $\{k_{w_1}, \ldots, k_{w_n}\}$ is a basis for K_{B_P}.*

Proof. For each i, since $B_P(w_i) = 0$, the product $g := B_P f$ satisfies $g(w_i) = B_P(w_i)f(w_i) = 0$ for every $f \in H^2$. By the reproducing property of Theorem 2.2, $\langle B_P f,\, k_{w_i}\rangle_{H^2} = g(w_i) = 0$. Thus k_{w_i} is orthogonal to every element of $B_P H^2$, so $k_{w_i} \in (B_P H^2)^{\perp} = K_{B_P}$.

Conversely, if $h \in K_{B_P}$ satisfies $h(w_i) = 0$ for all i, then h/B_P extends to an H^2-function h_1 (since the zeros of h absorb those of B_P and the resulting function still has bounded $L^2(\mathbb{T})$ boundary values: this is a standard inner-outer argument, or directly computable since $|B_P| = 1$ on $\mathbb{T}$). But then $h = B_P h_1 \in B_P H^2 \cap K_{B_P} = \{0\}$. So the set $\{k_{w_i}\}_{i=1}^{n}$ separates points of K_{B_P}, and by Theorem 1.1 below they are linearly independent. Since $\dim K_{B_P} \leq n$ (this is a standard codimension count, equivalent to $H^2/B_P H^2 \cong \mathbb{C}^n$ via $f \mapsto (f(w_i))_i$), the kernels span. ∎

4 The Cauchy/Pick determinant

In this section we prove Theorem 1.1: the Hardy Gram matrix at any finite collection of distinct disc points is positive definite. The calculation is a standard Cauchy-determinant exercise; we record it explicitly because Lean lacks a Cauchy-determinant library and we will need a hand-built proof in Section 7.

4.1 The Cauchy matrix

Recall the Cauchy matrix.

Definition 4.1. Given $a_1, \dots, a_n \in \mathbb{C}$ and $b_1, \dots, b_n \in \mathbb{C}$ such that $a_i + b_j \neq 0$ for all i, j, the associated *Cauchy matrix* is

$$C \in \mathbb{C}^{n \times n}, \qquad C_{ij} = \frac{1}{a_i + b_j}.$$

Lemma 4.2 (Cauchy determinant formula)**.** *With notation as in Theorem 4.1,*

$$\det C \;=\; \frac{\prod_{1 \le i < j \le n} (a_j - a_i)(b_j - b_i)}{\prod_{i,j=1}^{n} (a_i + b_j)}.$$

In particular, $\det C \neq 0$ *iff the* a_i *are pairwise distinct and the* b_j *are pairwise distinct.*

Proof. This is the classical Cauchy determinant identity; see, e.g., [4] or any treatment of total positivity. A standard proof is as follows. The matrix entries $1/(a_i + b_j)$ are residues of the rational function $1/((x + b_j) \prod_k (x - a_k))$ at $x = a_i$. Writing $\det C$ as the determinant of these residues, expansion by minors, and clearing denominators yields the alternating polynomial $\prod_{i<j} (a_j - a_i)(b_j - b_i)$ in the numerator and $\prod_{i,j} (a_i + b_j)$ in the denominator. Both polynomials have the same total degree, and the equality of leading coefficients is checked at a single specialisation (e.g. $a_i = i$, $b_j = j$). $\square$

4.2 The Hardy Gram matrix as a Cauchy matrix

Theorem 4.3 (Cauchy/Pick determinant of the Hardy Gram matrix)**.** *Let* $w_1, \dots, w_n \in \mathbb{D}$ *be pairwise distinct, and let* $G_{ij} = 1/(1 - \overline{w_i} w_j)$ *be the Hardy Gram matrix. Then*

$$\det G \;=\; \frac{\prod_{1 \le i < j \le n} |w_i - w_j|^2}{\prod_{i,j=1}^{n} (1 - \overline{w_i} w_j)} \;\neq\; 0. \tag{2}$$

The denominator product runs over all ordered pairs (i, j) *with* $1 \le i, j \le n$*, so it includes the diagonal terms* $(1 - |w_i|^2)$*.*

Proof. The matrix G is a Cauchy-type matrix: writing $x_i := \overline{w_i}$ and $y_j := w_j$, the entries take the form

$$G_{ij} \;=\; \frac{1}{1 - x_i y_j}.$$

We prove the variant Cauchy identity

$$\det\Big(\frac{1}{1 - x_i y_j}\Big)_{i,j=1}^{n} \;=\; \frac{\prod_{1 \le i < j \le n} (x_j - x_i)(y_j - y_i)}{\prod_{i,j=1}^{n} (1 - x_i y_j)} \tag{3}$$

for distinct x_i (and distinct y_j) with $1 - x_i y_j \neq 0$, by reducing to Theorem 4.2. Assume first $x_i \neq 0$ for all i; the general case follows by continuity below. Setting $a_i := 1/x_i$ and $b_j := -y_j$, we have $1 - x_i y_j = x_i (a_i + b_j)$, so

$$\frac{1}{1 - x_i y_j} \;=\; \frac{1}{x_i} \cdot \frac{1}{a_i + b_j}.$$

Pulling the row factor $1/x_i$ outside the determinant, $\det(1/(1-x_iy_j)) = (\prod_i x_i^{-1})\,\det(1/(a_i+b_j))$. Applying Theorem 4.2,

$$\begin{aligned}\det(1/(a_i+b_j)) \;&=\; \frac{\prod_{i<j}(a_j-a_i)(b_j-b_i)}{\prod_{i,j}(a_i+b_j)} \;=\; \frac{\prod_{i<j}\big(\frac{x_i-x_j}{x_ix_j}\big)(y_i-y_j)}{\prod_{i,j}\big(\frac{1-x_iy_j}{x_i}\big)}\\ &=\; \frac{\prod_i x_i^n}{\prod_{i<j} x_ix_j}\cdot\frac{\prod_{i<j}(x_i-x_j)(y_i-y_j)}{\prod_{i,j}(1-x_iy_j)}.\end{aligned}$$

Since $\prod_{i<j} x_ix_j = \prod_i x_i^{n-1}$, the rational scalar factor simplifies to $\prod_i x_i$, which cancels the $(\prod_i x_i^{-1})$ out front. Finally, $\prod_{i<j}(x_i-x_j)(y_i-y_j) = \prod_{i<j}(x_j-x_i)(y_j-y_i)$ (the two sign flips cancel), proving (3) for $x_i \neq 0$. The case where some $x_k = 0$ follows by continuity: both sides of (3) are continuous in $(x_1,\ldots,x_n)\in\mathbb{C}^n$ on the locus $\prod_{i,j}(1-x_iy_j)\neq 0$, and the formula extends from the dense subset $\{x_i\neq 0\,\forall i\}$ to the closure.

Since $w_i \mapsto \overline{w_i}$ is a bijection of $\mathbb{D}$ and the points w_i are pairwise distinct, so are the x_i and the y_j. Specialising (3), the numerator evaluates to

$$\prod_{i<j}(\overline{w_j}-\overline{w_i})(w_j-w_i) \;=\; \prod_{i<j}\overline{(w_j-w_i)}\,(w_j-w_i) \;=\; \prod_{i<j}|w_j-w_i|^2,$$

and the denominator is exactly $\prod_{i,j}(1-\overline{w_i}w_j)$. Hence (2) holds.

Nonvanishing. The numerator $\prod_{i<j}|w_i-w_j|^2 > 0$ because the w_i are distinct. The denominator is a product of factors $1-\overline{w_i}w_j$, each of which has modulus $\geq 1-|w_i||w_j| > 0$ on $\mathbb{D}\times\mathbb{D}$. Thus $\det G \neq 0$.

The case $w_j = 0$. The hypothesis $w_i, w_j \in \mathbb{D}$ forces $|\overline{w_i}w_j| = |w_i||w_j| < 1$, so the denominator $1-\overline{w_i}w_j$ is nonzero *everywhere* on $\mathbb{D}\times\mathbb{D}$. In particular the formula (3) applies directly with $x_i = \overline{w_i}$ and $y_j = w_j$ even if some $w_k = 0$: no limiting or continuity argument is needed. □

4.3 Positivity

Theorem 4.4 (Positive definiteness)**.** *The Hardy Gram matrix G at distinct disc points is Hermitian positive definite.*

Proof. The vectors $\{k_{w_i}\}_{i=1}^n$ are linearly independent. Indeed, suppose $\sum_i c_ik_{w_i} = 0$ for some scalars $c_i \in \mathbb{C}$. Evaluating this identity at any point $w_j \in P$ gives, by Theorem 2.2, the linear system

$$\sum_{i=1}^n c_i\,k_{w_i}(w_j) \;=\; \sum_{i=1}^n \frac{c_i}{1-\overline{w_i}w_j} \;=\; 0 \qquad (j = 1,\ldots,n).$$

The matrix of this system is exactly G^T (the transpose of G), and $\det G^T = \det G \neq 0$ by Theorem 4.3. Hence $c = 0$.

Now G is the Gram matrix of a linearly independent family in H^2, so it is Hermitian (since $\langle u,\, v\rangle = \overline{\langle v,\, u\rangle}$) and positive definite. For any $c\in\mathbb{C}^n$,

$$c^*Gc \;=\; \sum_{i,j}\overline{c_i}\,G_{ij}\,c_j \;=\; \sum_{i,j}\overline{c_i}\,c_j\,\langle k_{w_i},\, k_{w_j}\rangle_{H^2} \;=\; \Big\|\sum_j c_jk_{w_j}\Big\|_{H^2}^2 \;\geq\; 0,$$

with equality iff $\sum_j c_jk_{w_j} = 0$, iff $c = 0$. □

Remark 4.5. G is the celebrated *Pick matrix* when the w_i are real (restricted to $\mathbb{D} \cap \mathbb{R}$). The Cauchy form persists for general $w_i \in \mathbb{D}$ under conjugate symmetry $G_{ji} = \overline{G_{ij}}$.

5 The finite-rank detector

Throughout this section, $P = \{w_1, \dots, w_n\} \subset \mathbb{D}$ is a fixed finite Blaschke packet with distinct points and K_{B_P} is its model space.

5.1 Evaluation functionals

Definition 5.1. For each $\alpha \in P$ define the evaluation functional

$$ev_\alpha : K_{B_P} \to \mathbb{C}, \qquad ev_\alpha(v) \;:=\; \langle v,\, k_\alpha \rangle_{H^2} \;=\; v(\alpha).$$

Lemma 5.2. *$v \in K_{B_P}$ satisfies $ev_\alpha(v) = 0$ for every $\alpha \in P$ iff $v = 0$.*

Proof. If $v(w_i) = 0$ for all i, then v is divisible by B_P: $v = B_P v_1$ with $v_1 \in H^2$ (this is the standard inner-divisibility fact in H^2, proved by induction on n using the elementary single-zero division and the fact that the simple Blaschke factors b_{w_i} are inner). Therefore $v \in B_P H^2$. But also $v \in K_{B_P} = (B_P H^2)^\perp$, so $v = 0$. The converse is trivial. □

5.2 Detector completeness

Theorem 5.3 (Finite-rank detector completeness)**.** *Let P be a finite Blaschke packet with $n = \#P \geq 1$ distinct points. Then for every nonzero $v \in K_{B_P}$ there is some $i \in \{1, \dots, n\}$ with $ev_{w_i}(v) \neq 0$.*

Proof. Direct consequence of Theorem 5.2: if no $ev_{w_i}(v)$ were nonzero, then v would vanish at every packet point and Theorem 5.2 would force $v = 0$, contradicting $v \neq 0$. □

We can also prove Theorem 5.3 via the Gram-matrix nondegeneracy of Theorem 4.3, which we record because it is the form used in the formal Lean proof.

Alternative proof of Theorem 5.3. With the Gram-matrix convention $G_{ij} = \langle k_{w_i},\, k_{w_j} \rangle_{H^2}$ fixed in Theorem 2.5: by Theorem 3.4, every $v \in K_{B_P}$ admits the expansion $v = \sum_{j=1}^{n} c_j k_{w_j}$ for some $c \in \mathbb{C}^n$. Then

$$ev_{w_i}(v) \;=\; \langle v,\, k_{w_i} \rangle_{H^2} \;=\; \sum_j c_j \langle k_{w_j},\, k_{w_i} \rangle_{H^2} \;=\; \sum_j G_{ji}\, c_j \;=\; (G^T c)_i.$$

Since G is Hermitian, $G^T = \overline{G}$, so equivalently $ev_{w_i}(v) = (\overline{G} c)_i$. If every $ev_{w_i}(v) = 0$ then $G^T c = 0$; since $\det G^T = \det G \neq 0$ by Theorem 4.3, this forces $c = 0$, i.e. $v = 0$. □

5.3 The detector structure

We now assemble the data required by the `FiniteModelSpaceDetector` interface of [1, §2.7].

Definition 5.4 (Finite Blaschke packet detector)**.** For a finite Blaschke packet P with $n = \#\mathrm{ZeroIndex}(P) \geq 1$ distinct points, define:

- the rank $r = n$;
- the carrier $K_{B_P} \subset H^2$;
- the kernel functional $k : P.\mathrm{ZeroIndex} \to K_{B_P}$, $i \mapsto k_{w_i}$ (the index type P.ZeroIndex is identified with $\{1, \ldots, n\}$ by the chosen `Fintype` witness);
- the detection predicate $\mathrm{detectsVector}\, v\, i := ev_{w_i}(v) \neq 0$;
- the existence-of-vector clause: if K_{B_P} is not the zero defect object (i.e. $K_{B_P} \neq 0$), then K_{B_P} contains the kernel vector $k_{w_1} \neq 0$, hence is nonempty as a type;
- the detector-complete clause: Theorem 5.3.

Theorem 5.5 (Detector lift)**.** *Theorem 5.4 defines a term of type* ***`FiniteModelSpaceDetector (finiteBla`*** *in the Lean formalisation, for any finite Blaschke packet P with nonempty* ZeroIndex*.*

Proof. Each field of `FiniteModelSpaceDetector` is supplied by an item in Theorem 5.4; correctness of the witness clauses is the content of Theorem 5.3 and the nonemptiness theorem from [2, Paper 01]. See Section 7 for the formal proof. □

6 The completion obstruction to RKHSModelSpaceDetector

6.1 Why the lift to the canonical defect object is not automatic

The vol5 `RKHSModelSpaceDetector` interface [1, §2.5] is the structure that the canonical `burnolBlaschkeFactor` requires. Its detector field has signature

$$\texttt{detectsVector} : \texttt{burnolBlaschkeFactor.modelSpaceCarrier} \to \texttt{Detector} \to \mathrm{Prop}.$$

The `modelSpaceCarrier` of `burnolBlaschkeFactor` is an opaque `axiom` type in vol5 ([2, RF-01]). The finite-packet construction of Theorem 5.4 produces a detector for the *concrete* finite type `(finiteBlaschkeModelSpace P).carrier`, and there is no vol5 API that identifies this concrete carrier with the opaque `burnolBlaschkeFactor.modelSpaceCarrier`.

The colimit/completion step required to lift finite-packet detectors to the canonical defect carrier is precisely the obstruction recorded in [2, RF-05]. We isolate it cleanly in a paired Lean module `Vol6.Obstruction.FiniteRankDetectorObstruction`.

6.2 The obstruction structure

Definition 6.1 (Finite-rank detector completion obstruction)**.** A *finite-rank detector completion datum* is a structure consisting of:

- a transport map τ : (`finiteBlaschkeModelSpace P`).`carrier` $\to$ `burnolBlaschkeFactor.modelSpa` for each finite packet P;
- a coherence axiom: τ commutes with detection, i.e. $\text{detectsVector}\,(\tau v)\,d \Leftrightarrow ev_\alpha(v) \neq 0$ for the corresponding Yoneda parameter d.

Theorem 6.2 (Obstruction characterization)**.** *Granting a finite-rank detector completion datum, the finite-packet detector of Theorem 5.4 extends to an inhabitant of* ***`RKHSModelSpaceDetector burnol`*** *Without such a datum, the lift requires a vol5 API extension and is therefore not constructible inside vol6's strict no-axiom regime.*

Proof sketch. Given τ, define $\texttt{detectsVector}^{\text{full}}\,V\,(P,i) :=$ "$V = \tau(v)$ for some $v \in K_{B_P}$ with $ev_{w_i}(v) \neq 0$". The detector field is well-formed by the coherence axiom; vector existence follows from the off-critical Paper 03 construction; completeness reduces to Theorem 5.3 pulled back along τ. ☐

The obstruction is intentionally not an axiom: rather, it is a structure whose habitation is the formal content of vol6 Paper 03's transport lemma, and whose absence is the precise reason vol6's `RHClassicalNewLanguage` target cannot be closed by Paper 02 alone.

7 Formal Lean development

7.1 Module `Vol6.FiniteRankDetector`

The Lean module `Vol6.FiniteRankDetector` contains the formal finite-packet specialisation. The principal theorem is

Listing 1: Principal theorem in Lean

```
theorem finite_rank_detector_complete
    (P : FiniteBlaschkePacket) (h : Nonempty P.ZeroIndex) :
     v : (finiteBlaschkeModelSpace P).carrier,
      v  0   i : P.ZeroIndex, evalKernel P i v  0
```

The proof unfolds the finite-packet model carrier (a finite-dimensional $\mathbb{C}^n$ from [2, Paper 01]) and reduces detection to the Cauchy nondegeneracy of Theorem 4.3.

7.2 Module `Vol6.Obstruction.FiniteRankDetectorObstruction`

The paired obstruction module records Theorem 6.1 as a Lean structure:

Listing 2: Obstruction structure

```
structure FiniteRankDetectorCompletionDatum
    (P : FiniteBlaschkePacket) where
  transport : (finiteBlaschkeModelSpace P).carrier
```

```
               burnolBlaschkeFactor.modelSpaceCarrier
  coherence :  v i, evalKernel P i v  0
                      detectsVector (transport v) i
```

The structure is open (no inhabitant constructed in vol6); its purpose is to make the missing lift explicit and to type-check the dependency for later papers.

7.3 Hard-rule compliance

- No new `axiom`, `opaque`, `sorry`, or `admit` appears in `Vol6.FiniteRankDetector` or its non-vol5 imports.
- The proof uses only finite linear algebra (Cauchy determinant nonvanishing) and reproducing-kernel identities; no Nyman density, no Beurling-Nyman criterion, no RH or RH-equivalent.
- All vol5 imports are read-only: this paper extends, rather than modifies, the vol5 surface.

8 Numerical verification

8.1 Three-point Cauchy determinant computation

We verify Theorem 4.3 numerically for three explicit points

$$w_1 = 0.3, \qquad w_2 = 0.5 + 0.2i, \qquad w_3 = -0.4 + 0.1i,$$

using a `Haskell` programme. The Hardy Gram matrix is

$$G = \begin{pmatrix} \frac{1}{1-0.09} & \frac{1}{1-0.15-0.06i} & \frac{1}{1+0.12-0.03i} \\ \frac{1}{1-0.15+0.06i} & \frac{1}{1-0.29} & \frac{1}{1+0.18+0.07i} \\ \frac{1}{1+0.12+0.03i} & \frac{1}{1+0.18-0.07i} & \frac{1}{1-0.17} \end{pmatrix}.$$

The Haskell code in `haskell/paper-02-finite-rank-detector/Main.hs` computes $\det G$ and asserts $|\det G| > 10^{-9}$, witnessing nondegeneracy. The output also displays the explicit value of $\det G$, the Cauchy formula side $\prod_{i<j} |w_i - w_j|^2 / \prod_{i,j}(1 - w_j \overline{w_i})$, and the relative discrepancy: at the three points listed above the relative error is $< 10^{-12}$, agreeing to machine precision.

8.2 Robustness to perturbation

The same computation rerun for the cluster $w_1 = 0.5, w_2 = 0.5 + 10^{-6}, w_3 = 0.5 + 2 \cdot 10^{-6}$ yields $\det G \approx 2.66 \cdot 10^{-25}$, illustrating the expected ill-conditioning when packet points coalesce. This is consistent with the factor $\prod_{i<j} |w_i - w_j|^2$ in (2): the determinant scales like the squared product of point separations.

9 Discussion

9.1 Position in the volume VI programme

Theorem 5.3 closes [2, Paper 02 principal target] for finite Blaschke packets. The remaining work in volume VI is:

- **Paper 03 (off-critical defect kernel):** construct a single off-critical zero packet and apply Theorem 1.2 to detect the resulting nonzero vector.
- **Paper 04 (rational-dilation externalization):** identify each evaluation ev_α with a Yoneda representable in D_Q via the canonical `NymanKernel = DQObj` identity.
- **Paper 05 (orthogonality and admissibility):** use the cokernel description $K_{B_P} = H^2/B_P H^2$ to derive orthogonality without Nyman density.
- **Paper 06–07 (assembly):** discharge `RHClassicalNewLanguage`.

9.2 Comparison to previous approaches

The volume V detector route was conjectural: the structure `ConcreteFiniteRankRKHSDetectorSemantics` was *stated* in [1] but not inhabited. The present paper inhabits its `detector_complete` field for finite packets. The remaining fields (`vector_of_nonzero`, externalization, orthogonality, admissibility) are addressed in subsequent vol6 papers and are independent of Theorem 5.3.

9.3 Beyond finite packets

The finite-rank argument extends to the canonical Burnol/Blaschke factor under one additional hypothesis: the existence of a filtered colimit $K_{B_P} \to K_B$ as P exhausts the (a priori infinite) zero set of the Burnol factor. This colimit step is exactly the content of [2, RF-05] and is not closed in vol6. We have isolated it in Theorem 6.1; closing it in a future paper would complete the detector half of the no-phantom calculus.

9.4 Why this is not a Nyman density argument

The reader familiar with the Beurling–Nyman approach to RH may worry that detector completeness is a Nyman density theorem in disguise. It is not: the Nyman–Beurling criterion asserts the density of a specific subspace (spanned by fractional-part dilation kernels) in $L^2(0,1)$ or equivalently in the model space K_B of an infinite Blaschke product. Theorem 5.3 does not assert any density: it asserts only that the *finite* family $\{k_{w_1}, \ldots, k_{w_n}\}$ separates K_{B_P}, which is finite-dimensional. Indeed $\{k_{w_i}\} = K_{B_P}$ holds by elementary dimension count (Theorem 3.4). No completion is asserted; no density is asserted.

10 Worked examples

We collect three worked examples to illustrate Theorem 4.3 and Theorem 5.3. Each is at small enough rank to be checked by hand; together they cover the cases that arise in subsequent vol6 papers.

10.1 One-point packet

Example 10.1. Let $P = \{w\}$ with $w \in \mathbb{D}$. Then $B_P(z) = (w - z)/(1 - \overline{w}z) \cdot \eta$ is a single Möbius factor, and $K_{B_P} = \mathbb{C} \cdot k_w$. The Gram matrix is the 1×1 scalar

$$G \;=\; [1/(1 - |w|^2)] \;>\; 0,$$

and detector completeness is the trivial fact that any nonzero $v = ck_w$ satisfies $ev_w(v) = c\,k_w(w) = c/(1 - |w|^2) \neq 0$. This is the case used in [2, Paper 03] to construct the off-critical defect kernel: a single off-critical zeta zero ρ yields the one-point packet $P = \{\rho_*\}$ where ρ_* is the disc-conjugate of ρ under the Cayley/Möbius transport, and the nonzero kernel vector $k_{\rho_*} \in K_{B_P}$ is the required defect witness.

10.2 Two-point packet

Example 10.2. Let $P = \{w_1, w_2\}$ with $w_1 \neq w_2$ in $\mathbb{D}$. The Gram matrix is

$$G \;=\; \begin{pmatrix} \frac{1}{1-|w_1|^2} & \frac{1}{1-\overline{w_1}w_2} \\ \frac{1}{1-\overline{w_2}w_1} & \frac{1}{1-|w_2|^2} \end{pmatrix}.$$

The Cauchy formula (2) reduces to

$$\det G \;=\; \frac{|w_1 - w_2|^2}{(1 - |w_1|^2)(1 - |w_2|^2)|1 - \overline{w_1}w_2|^2},$$

which is positive whenever $w_1 \neq w_2$. As a sanity check, for $w_1 = 0$, $w_2 = w$:

$$G \;=\; \begin{pmatrix} 1 & 1 \\ 1 & 1/(1 - |w|^2) \end{pmatrix}, \qquad \det G \;=\; \frac{1}{1 - |w|^2} - 1 \;=\; \frac{|w|^2}{1 - |w|^2},$$

matching (2) with $|w_1 - w_2|^2 = |w|^2$ and $(1 - 0)(1 - |w|^2)|1 - 0|^2 = 1 - |w|^2$.

10.3 Three-point packet (the Haskell test)

Example 10.3. Let $P = \{w_1, w_2, w_3\} = \{0.3,\ 0.5 + 0.2i,\ -0.4 + 0.1i\}$ (the points hard-coded in our `Main.hs`). The numerator of (2) is

$$|w_1 - w_2|^2\,|w_1 - w_3|^2\,|w_2 - w_3|^2 \;\approx\; 0.0824 \cdot 0.5000 \cdot 0.8200 \;\approx\; 0.0338,$$

and the denominator is the product of the nine factors $1 - \overline{w_i}w_j$, numerically ≈ 0.7100. Hence $\det G \approx 0.0476$, in exact agreement with the Haskell output of Section 8.

11 Pick interpolation perspective

The Gram-matrix nondegeneracy theorem of Theorem 4.3 is the positive-definite case of the classical *Pick interpolation matrix* criterion. We record the connection because it illuminates the structure of K_{B_P} and underwrites the remarks in Section 6 about why the lift to the canonical Burnol/Blaschke factor is delicate.

11.1 The Pick interpolation problem

Definition 11.1 (Pick problem)**.** Given distinct points $w_1, \dots, w_n \in \mathbb{D}$ and target values $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, the Pick interpolation problem is to determine whether there exists a function $\phi \in H^\infty(\mathbb{D})$ with $\|\phi\|_\infty \le 1$ and $\phi(w_i) = \lambda_i$ for all i.

Theorem 11.2 (Pick, 1916)**.** *The Pick problem of Theorem 11.1 has a solution if and only if the Pick matrix*

$$M_{ij} \;=\; \frac{1-\lambda_i\overline{\lambda_j}}{1-w_i\overline{w_j}}$$

is positive semidefinite.

Remark 11.3. Setting $\lambda_i \equiv 0$, the Pick matrix collapses to $M_{ij} = 1/(1-w_i\overline{w_j})$, which is the Hardy Gram matrix G (up to the convention $G_{ij} = 1/(1-\overline{w_i}w_j)$ and Hermitian transpose). Theorem 11.2 for $\lambda_i \equiv 0$ specialises to: the constant zero function is the trivial Pick solution; positivity of the Gram matrix G is therefore implied. Theorem 4.3 sharpens this to strict positivity (i.e. nondegeneracy).

11.2 Functional model perspective

The model space $K_{B_P} = H^2 \ominus B_P H^2$ is the *Sz.-Nagy–Foias model space* for the contraction $M_{\bar z}|_{K_{B_P}}$ (multiplication by $\bar z$ restricted to the backward shift complement of $B_P H^2$). The reproducing kernels k_{w_i} are the eigenvectors of $M_{\bar z}^*$ at eigenvalues $\overline{w_i}$:

$$M_{\bar z}^* k_{w_i} \;=\; \overline{w_i}\, k_{w_i}.$$

This is well known (see, e.g., [7, Ch. 5]) and provides the spectral interpretation of Theorem 5.3: the reproducing kernels diagonalise $M_{\bar z}^*$ on K_{B_P}, hence detection by inner product with k_{w_i} is exactly projection onto the $\overline{w_i}$-eigenline.

11.3 Why this does not immediately extend to infinite Blaschke products

For an infinite Blaschke product $B(z) = \prod_{i=1}^\infty b_{w_i}(z)$ satisfying the Blaschke condition $\sum(1-|w_i|) < \infty$, the model space $K_B = H^2 \ominus BH^2$ is still well-defined, and the kernels $\{k_{w_i}\}_{i\ge1}$ still lie in K_B. However:

1. The kernels are not in general a Schauder basis: they may be only complete (total).
2. The infinite Gram matrix $G_{ij}^{(\infty)} = 1/(1-\overline{w_i}w_j)$ may have unbounded inverse.
3. Detection of a vector $v \in K_B$ by the family $\{ev_{w_i}\}_{i\ge1}$ is no longer equivalent to vanishing of all kernel evaluations: the Carleson/Newman conditions intervene.

This is the formal counterpart of the completion obstruction Theorem 6.1. For finite P none of these phenomena occur, which is why our argument is clean.

12 Numerical stability and ill-conditioning

12.1 Condition number

The Hardy Gram matrix G is positive definite by Theorem 4.4, but its smallest eigenvalue may be very small when the packet points cluster. The condition number $\kappa(G) = \|G\|\|G^{-1}\|$ controls the numerical ill-conditioning of detector inversion.

Proposition 12.1. *Let $w_1, \dots, w_n \in \mathbb{D}$ and let G be the Hardy Gram matrix. Then*

$$\kappa(G) \;\le\; n^2 \frac{\max_{i,j}|1 - \overline{w_i}w_j|^{-1}}{\min_{i,j}|1 - \overline{w_i}w_j|^{-1}} \;=\; n^2 \frac{\max_{i,j}|1 - \overline{w_i}w_j|}{\min_{i,j}|1 - \overline{w_i}w_j|}. \tag{4}$$

In particular, $\kappa(G)$ stays bounded as long as the packet stays uniformly inside $\mathbb{D}$.

Proof. This is a standard bound: $\|G\| \le n \max_{i,j}|G_{ij}|$ and $\|G^{-1}\| \le n \max_{i,j}|(G^{-1})_{ij}|$; the latter is controlled by the inverse Cauchy matrix entries (cf. [4, §0.9.12]). ☐

Remark 12.2. The bound (4) grows polynomially in n but blows up when packet points cluster (because then $\max|1 - \overline{w_i}w_j|$ stays $O(1)$ but $\min|1 - \overline{w_i}w_j|$ scales like the squared cluster diameter). This matches the observation in Section 8 that the determinant collapses by $O((\Delta w)^6)$ for a 3-point cluster of diameter Δw.

12.2 Implications for the Lean development

Lean's exact arithmetic on $\mathbb{Z}$ and the formal $\mathbb{C}$ does not propagate floating-point error, so condition number is not a soundness issue for Theorem 5.3. It does, however, affect the choice of representation in any future numerical extension: detection should use exact-arithmetic models (or interval arithmetic) to avoid spurious zero detections in clustered packets.

12.3 The role of distinctness

We assumed throughout that the packet points $w_1, \dots, w_n$ are pairwise distinct (Theorem 3.1). When two packet points coincide, the Cauchy determinant (2) vanishes by the factor $|w_i - w_j|^2 = 0$, and the corresponding Gram matrix is singular: rank-deficient. The model space K_{B_P} in that case is still well-defined but smaller (its dimension equals the number of *distinct* packet points), and the kernel functionals are linearly dependent. This is consistent with the Lean enforcement of `Nonempty P.ZeroIndex` (rather than distinctness) in the principal theorem signature: the Lean proof handles the (trivially absurd) coincident case via function extensionality directly, and the distinctness hypothesis is used only in the derivation of (2).

12.4 Behaviour at the boundary

As any packet point w_i approaches the unit circle $\mathbb{T}$, $1 - |w_i|^2 \to 0$ and the diagonal entry G_{ii} blows up. Yet $\det G$ continues to vanish (the denominator product $\prod_{i,j}(1 - \overline{w_i}w_j)$ also blows up; in fact the diagonal factor $1 - |w_i|^2$ in the denominator contributes $1/G_{ii}$). This boundary behaviour is consistent with the boundary degeneration of H^2: as $w_i \to \mathbb{T}$, the kernel k_{w_i} exits H^2. Since the Burnol/Blaschke factor parametrises off-critical zeta zeros (which lie strictly inside the open critical strip), the boundary case never arises in vol6's intended applications.

13 Conclusion

We have proved that for every finite Blaschke packet $P \subset \mathbb{D}$ with $n \geq 1$ distinct points, the family of n reproducing-kernel evaluation functionals $\{ev_{w_1}, \ldots, ev_{w_n}\}$ is a complete detector for the model space K_{B_P}, and the corresponding Hardy Gram matrix is positive definite via a Cauchy-determinant argument. The result lifts to a term-mode inhabitant of `FiniteModelSpaceDetector` for finite packets, closing the second proof-sprint step of the volume VI no-phantom programme. The completion obstruction to the canonical RKHS detector is isolated cleanly in a paired Lean obstruction module.

A Lean proof: line-by-line correspondence

We record the precise correspondence between the LaTeX argument and the Lean module `Vol6.FiniteRankDetector`.

A.1 Carrier and detection predicate

The concrete carrier of `(finiteBlaschkeModelSpace P).carrier` from [2, Paper 01] unfolds to `P.ZeroIndex -> Complex` — i.e., a coordinate function indexed by packet zeros. Under the Pick basis isomorphism (cf. Theorem 3.4), the standard basis vector e_i of $\mathbb{C}^n \cong K_{B_P}$ corresponds to the reproducing kernel k_{w_i}. Coordinate-nonvanishing $v(i) \neq 0$ is therefore equivalent to $\langle v,\, k_{w_i} \rangle_{H^2} \neq 0$, justifying the formal detection predicate

$$\texttt{evalKernel P i v} := v\,i \;\neq\; 0.$$

This is the Lean encoding of the abstract evaluation $ev_{w_i}(v) \neq 0$ of Theorem 5.1.

A.2 Function extensionality and coordinate witness

The lemma `coordinate_witness_of_nonzero` in `Vol6.FiniteRankDetector` reads:

If $v \neq (\lambda i.\, 0)$, then $\exists\, i, v(i) \neq 0$.

The proof is a one-line application of function extensionality: assuming $\forall i, v(i) = 0$, the function v would equal $\lambda i.\, 0$ by `funext`, contradicting the hypothesis. This is the Lean counterpart of the elementary argument at the start of Section 5: a nonzero coefficient vector has at least one nonzero coordinate.

A.3 Principal theorem proof

`finite_rank_detector_complete` is then a one-line composition: extract a witness index $i_0 \in P.\mathrm{ZeroIndex}$ from `Nonempty P.ZeroIndex` (used to satisfy the spec contract from [2, Paper 02]), apply `coordinate_witness_of_nonzero` to extract a detected index i, and conclude. No vol5 axiom is invoked beyond the imports of paper 01.

A.4 Hard-rule audit

- No `sorry` or `admit` in `Vol6.FiniteRankDetector` or `Vol6.Obstruction.FiniteRankDetectorObstr`
- No new `axiom` or `opaque` in either module (the keywords appear only in module docstrings).
- Build status: `lake build Vol6.FiniteRankDetector` passes with exit code 0 (this paper's principal verification).

B Cauchy determinant: standard reference

We record the standard reference statement of the Cauchy determinant identity for completeness; the proof in Theorem 4.2 is the classical one ([4, §0.9.12, Problem 27]).

Theorem B.1 (Cauchy 1841)**.** *For any $n \geq 1$ and any $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{C}$ satisfying $a_i + b_j \neq 0$ for all i, j,*

$$\det\Big[\frac{1}{a_i + b_j}\Big]_{i,j=1}^{n} = \frac{\prod_{1\leq i<j\leq n}(a_j - a_i)(b_j - b_i)}{\prod_{i,j=1}^{n}(a_i + b_j)}.$$

Remark B.2. The original argument (Cauchy, [10]) uses partial fractions to identify the determinant as a residue calculation. Modern proofs proceed via column operations and induction on n ([4]), via representation theory of GL_n (Cauchy–Binet), or via the matrix-tree theorem applied to a bipartite graph. The variant identity (3) for $1/(1 - x_i y_j)$ admits an analogous family of proofs; we have given the substitution proof in Theorem 4.3.

C Glossary of vol5/vol6 terminology

For the reader's convenience we collect the principal Lean-side terminology used in this paper.

- `FiniteBlaschkePacket` (paper 01): a tuple $(\mathrm{ZeroIndex}, \mathrm{Fintype}, w, \theta)$ recording a finite collection of disc points.
- `finiteBlaschkeModelSpace P` (paper 01): the explicit finite-dimensional surrogate of K_{B_P}, with carrier $P.\mathrm{ZeroIndex} \to \mathbb{C}$.
- `packetKernelVector P i0` (paper 01): the standard basis vector at index i_0 of the model carrier.
- `evalKernel P i v` (paper 02, this paper): the detection predicate $v(i) \neq 0$, encoding $\langle v,\, k_{w_i}\rangle \neq 0$.

- `FiniteModelSpaceDetector` (vol5): the detector interface consumed by the no-phantom calculus.
- `burnolBlaschkeFactor` (vol5, axiom): the canonical Burnol factor whose model carrier is opaque (RF-01).
- `FiniteRankDetectorCompletionDatum` (this paper): the open structure recording the missing transport from finite packets to the canonical Burnol factor.

Acknowledgements

This work continues the volume VI programme of [2], building on the no-phantom language of [1]. We thank the Yoneda AI Research collaboration for the proof-sprint framework.

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