Fractional-Part Kernels as D_Q Representables

Rational-Dilation Externalization of Finite-Rank RKHS Detectors

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Abstract

We discharge the externalization component of Target 2 of the *No-Phantom Language* program by exhibiting, for any finite-rank reproducing-kernel-Hilbert-space (RKHS) detector on the Burnol/Blaschke defect object, a corresponding family of *rational-dilation Yoneda representables* in the presheaf category $\mathbf{PSh}(D_Q)$. The key identification, available from Vol5's `NoPhantomConcreteAtoms` surface, is the definitional identity D_Q-objects $\equiv \mathsf{NK}$ (Nyman kernels), which makes every rational-dilation generator already a Nyman representable, where a presheaf F is *Nyman representable* when $F = \mathrm{y}(X)$ for some Nyman kernel $X \in \mathsf{NK}$. Combined with the Mellin/multiplicative description of the Hardy space H^2, each kernel-evaluation functional $\mathrm{ev}_\alpha : K_B \to \mathbb{C}$ is identified with the fractional-part atom $D_{Q(\alpha)}$, where the rational dilation parameter $Q(\alpha) \in (0, 1] \cap \mathbb{Q}$ is induced by the multiplicative coordinate of α. We prove, in the Lean 4 formalization, the principal theorem

`burnol_blaschke_detector_externalization : RKHSDetectorExternalizationToRationalRepresentab`

relative to a vol5-exposed introduction rule for the opaque predicate `DefectDetectedByRationalDilationRe` and we characterize precisely the missing introduction rule in an *obstruction module* when the rule is not exposed. The mathematics is purely categorical/Hardy-theoretic: no Nyman density, no Beurling–Nyman criterion, no assumption of the Riemann hypothesis, and no use of any RH-equivalent.

Contents

1 Introduction

1.1 Motivation: the No-Phantom Language program

The Vol5 *No-Phantom Language* program reduces classical RH to two precise non-classical payloads:

(a) `OffCriticalZeroDefectKernel` – every off-critical zero of the Riemann zeta function creates a non-zero vector in the Burnol/Blaschke model space $K_B = H^2/BH^2$;

(b) `YonedaBlaschkeDetectorCalculus` – the Burnol/Blaschke defect carries a complete RKHS detector calculus, externalizable to rational-dilation Yoneda representables, with quotient orthogonality and admissibility.

The master conditional theorem

`RH_classical_of_no_phantom_language_breakthrough`

combines (a) and (b) into `RH_classical`.

Vol5 leaves three components of the second payload as named obligations for Vol6: (i) detector completeness, (ii) rational-dilation *externalization*, and (iii) quotient orthogonality plus admissibility. Paper 04 of the Vol6 sprint, the present paper, treats component (ii).

1.2 What externalization means

Externalization is the categorical bridge between two pictures of the same data:

Internal picture (RKHS detector). A finite family of *kernel evaluation functionals* $\{\mathrm{ev}_{\alpha_i}\}_{i=1}^{r}$ on the model space K_B which *detects* non-zero vectors:

$$v \in K_B \setminus \{0\} \implies \exists\, i \in \{1, \ldots, r\} \text{ with } \mathrm{ev}_{\alpha_i}(v) \neq 0.$$

This is the internal RKHS / model-space side of the bridge.

External picture (rational-dilation representable). For each detector index i, a presheaf $F_i \in \mathbf{PSh}(D_Q)$ that is equal to a Yoneda representable:

$$F_i = \mathrm{y}\big(\mathsf{obj}_Q(\theta_i)\big), \qquad \theta_i \in \mathbb{Q} \cap (0, 1],$$

where D_Q is the rational-dilation test category and $\mathsf{obj}_Q \colon \mathbb{Q} \cap (0, 1] \to D_Q$ is the canonical embedding. The *detected_of_detector* compatibility says that any internal detection of v by ev_{α_i} can be *seen* by the externalized presheaf F_i probing the defect object.

The point of the bridge is that the external side connects the analytic detector to the categorical structure used by the `NoPhantomBlaschkeDefectRigidity` principle to force a contradiction with internal Blaschke triviality.

1.3 Outline of the paper

Section 2 reviews the categorical framework: the rational-dilation test category D_Q, the Yoneda embedding, Nyman kernels, and the Burnol/Blaschke defect object. **??** recalls the fractional-part atoms and the Mellin transport between the Hardy and multiplicative pictures. **??** reviews the finite-rank RKHS detector machinery (Paper 02 of the sprint). Section 3 states and proves the principal externalization theorem in two flavours: the conditional version parameterised by an introduction rule for the opaque `DefectDetectedByRationalDilationReprese` and the *trivial-detector* version that produces a no-argument inhabitant of the externalization structure. Section 4 characterises precisely which vol5 surface extension would close the externalization unconditionally. Section 5 traces the Lean 4 formalization. Section 7 discusses the Haskell numerical demonstration: computation of fractional-part kernels and Mellin pairings against finite Blaschke packets. Section 8 collects the formal results. Section 9 discusses implications for the rest of the Vol6 sprint.

1.4 Hard rules

In keeping with the Vol6 charter:

- No `sorry`, `admit`, new `axiom`, or new `opaque` appears in this paper's Lean module.
- No use of Nyman density, the Beurling–Nyman equivalence, RH, or any RH-equivalent.
- Vol5 is read-only.

2 Mathematical Framework

2.1 The rational-dilation test category D_Q

Definition 2.1 (Rational dilation parameters)**.** A *rational dilation parameter* is a real number θ with

$$0 < \theta \leq 1, \qquad \theta \in \mathbb{Q}.$$

The set of rational dilation parameters is $\Theta := (0, 1] \cap \mathbb{Q}$.

Definition 2.2 (Test category D_Q)**.** The *rational-dilation test category* D_Q is the category whose

- objects form the set NK of *Nyman kernels* in vol5 language, a fixed type used to tag rational-dilation generators, identified definitionally with the type `DQObj` (as exposed in vol5) of D_Q-objects, with canonical map $\mathsf{obj}_Q \colon \Theta \to \mathsf{NK}$ from rational dilation parameters;
- morphisms are recorded as the (set-valued) family $D_Q(X, Y)$ for $X, Y \in \mathsf{NK}$, equipped with:
  - an identity morphism $\mathsf{id}_X \in D_Q(X, X)$ for every object X,
  - an associative composition $\circ \colon D_Q(X, Y) \times D_Q(Y, Z) \to D_Q(X, Z)$ satisfying the standard categorical axioms (associativity, identity laws).

The vol5 surface (`YonedaCore.lean`) records the morphism API at exactly this granularity: an axiomatic hom-type `DQHom`$\colon D_Q \to D_Q \to$ Type, identity arrows `DQ_id`$\colon (X \colon D_Q) \to$ `DQHom` X X, and composition `DQ_comp`$\colon \forall X, Y, Z,$ `DQHom` X $Y \to$ `DQHom` Y $Z \to$ `DQHom` X Z. This is the minimal categorical structure required by the Yoneda machinery in theorems 2.5 and 2.6 below.

Remark 2.3 (On the discrete vs. non-discrete reading)*.* The presheaf-theoretic content of this paper does not depend on whether D_Q is taken to be a discrete category (i.e. $D_Q(X, Y) = \emptyset$ for $X \neq Y$ and $D_Q(X, X) = \{\mathsf{id}_X\}$) or a richer category equipped with non-trivial dilation morphisms. What matters for our externalization theorem is only that D_Q satisfies the standard categorical axioms recorded in theorem 2.2 so that the Yoneda functor of theorem 2.6 is well-defined. In the vol5 formalization the morphism type `DQHom` is left as an axiom precisely so that the formal proofs do not commit to either reading.

Remark 2.4 (Vol5 surface identity)*.* Vol5's `YonedaCore.lean` records the type-level identity `DQObj` $\equiv$ `NymanKernel` as `abbrev DQObj : Type := NymanKernel`. This collapses the test category and the Nyman generator type in the formalization, while still keeping them notationally distinct in mathematics: every D_Q object *is* a Nyman kernel tag and conversely. This is the key to externalization: any $F = \mathrm{y}(X)$ for $X \in D_Q$ is automatically a Nyman representable.

2.2 Yoneda presheaves on D_Q

Definition 2.5 (D_Q-presheaf)**.** A D_Q*-presheaf* is a pair $(F_{\mathrm{obj}}, F_{\mathrm{map}})$ consisting of:

- an object map $F_{\mathrm{obj}} \colon D_Q \to \mathbf{Set}$,
- a contravariant arrow map $F_{\mathrm{map}} \colon D_Q(U, V) \to (F_{\mathrm{obj}}(V) \to F_{\mathrm{obj}}(U))$

recording the API of a presheaf.

Definition 2.6 (Yoneda functor)**.** For $X \in D_Q$, the *Yoneda presheaf* $\mathrm{y}(X) \in \mathbf{PSh}(D_Q)$ is

$$\mathrm{y}(X)_{\mathrm{obj}}(U) := D_Q(U, X), \qquad \mathrm{y}(X)_{\mathrm{map}}(f)(g) := f \circ g.$$

A presheaf F is *representable* if there exists $X \in D_Q$ with $F = \mathrm{y}(X)$.

Theorem 2.7 (Yoneda is representable)**.** *For every $X \in D_Q$, the presheaf* $\mathrm{y}(X)$ *is representable. In fact,* $\mathrm{y}(X)$ *is* by definition *the representable presheaf attached to X: the witness object is X, and the required equality* $\mathrm{y}(X) = \mathrm{y}(X)$ *is reflexivity.*

This statement, which appears tautological at the categorical level because it amounts to "a representable functor is representable", is recorded as a theorem in the formalization because the predicate `Representable` *in theorem 2.6 is stated using existential quantification over an unknown witness object. The theorem unfolds the existential against the witness X and reduces the equation to a definitional one, which Lean discharges by* `rfl`*.*

Proof. Choose X as the witness in the existential quantifier of `Representable` (theorem 2.6); in the Lean formalization this is recorded by `Exists.intro` X. The remaining equation $\mathrm{y}(X) = \mathrm{y}(X)$ then holds by reflexivity of equality, which Lean discharges with `rfl` once the witness X has been introduced. □

2.3 The Burnol/Blaschke defect object

Definition 2.8 (Hardy submodule and ambient)**.** The ambient Hilbert space is the Hardy space H^2 on the disk $\mathbb{D}$. The *Burnol/Blaschke submodule* $BH^2 \subseteq H^2$ is the closed multiplication image of a Blaschke product B. In Vol5 these are introduced as opaque axioms `HardySpaceH2` and `HardySubmodule` together with a distinguished `blaschkeHardyImage : HardySubmodu`

Definition 2.9 (Blaschke defect object)**.** The *Blaschke defect object* is the structure

$$K_B = (\texttt{hardyCarrier} = BH^2,\ \texttt{modelCarrier} = H^2/BH^2).$$

The model carrier represents the quotient/cokernel/orthogonal complement of BH^2 in H^2. In Vol5 it is recorded as the canonical `blaschkeDefectObject` via `burnolBlaschkeFactor.modelSpaceCarri`

2.4 Fractional-part kernels and the Mellin picture

The connection between K_B and D_Q goes through the Mellin transform, which transports the multiplicative coordinate $x \in (0, 1]$ to the additive coordinate $\sigma + i\tau$ on the critical strip and identifies H^2 on the disk with $L^2((0, 1], dx/x)$ on a corresponding fundamental domain.

Definition 2.10 (Fractional-part atom)**.** For $\theta \in \Theta$, the *fractional-part atom* is the function

$$\rho_\theta(x) := \{\theta/x\}, \qquad x \in (0, 1],$$

where $\{y\} = y - \lfloor y \rfloor$ is the fractional part. We write $A_\theta := \rho_\theta \in L^2((0, 1])$ when convergence is clear.

Remark 2.11. The atoms A_θ are the original Beurling–Nyman generators of the linear span used in the classical density approach. In our *non-density* program we never invoke their density; we use only the data $\theta \mapsto A_\theta$ as a parametrization of points in NK.

Example 2.12 (Computation of $A_{1/3}$)**.** For $\theta = 1/3$, the atom $A_{1/3}(x) = \{1/(3x)\}$. At $x = 1/2$, $A_{1/3}(1/2) = \{2/3\} = 2/3$. At $x = 1$, $A_{1/3}(1) = \{1/3\} = 1/3$. At $x = 1/4$, $A_{1/3}(1/4) = \{4/3\} = 1/3$. The function has discontinuities at the points $x = 1/(3k)$ for $k = 1, 2, 3, \ldots$, where $\{1/(3x)\}$ jumps from 1 down to 0. Numerically, the atom $A_{1/3}$ on a 16-point grid takes the values listed in the accompanying Haskell output.

Example 2.13 (Two atoms with the same D_Q object)**.** The map $\theta \mapsto A_\theta$ is generically injective on Θ (rational θ values give distinct sawtooth functions), but the D_Q object $\mathsf{obj}_Q(\theta) \in \mathsf{NK}$ may identify multiple values of θ when the morphism structure is non-trivial. In the discrete reading of theorem 2.2 this collapse does not happen and obj_Q is injective; in a richer reading (with non-trivial dilation morphisms), it may. The externalization theorem of this paper does not depend on either reading.

2.5 The kernel evaluation functional in Hardy coordinates

Lemma 2.14 (Reproducing kernel in H^2)**.** *For $\alpha \in \mathbb{D}$, the reproducing kernel of H^2 at α is the Cauchy kernel*

$$k_\alpha^{H^2}(z) = \frac{1}{1 - z\,\overline{\alpha}}.$$

The reproducing kernel of the model space $K_B = H^2/BH^2$ at a packet zero α is

$$k_\alpha^{K_B}(z) = \frac{1 - B(z)\,\overline{B(\alpha)}}{1 - z\,\overline{\alpha}},$$

which simplifies to $k_\alpha^{H^2}(z)$ when $B(\alpha) = 0$ (i.e. when α is itself a packet zero).

Proof sketch. The first formula is classical Hardy theory. The second is the de Branges–Rovnyak formula for the reproducing kernel of the quotient H^2/BH^2 as a sub-Hilbert space of H^2. Both formulas are standard and are recorded here only to fix notation; they are not used in the Lean formalization (which works at the higher level of the abstract defect object). □

Definition 2.15 (Reproducing-kernel evaluation functional)**.** Let $\alpha \in \mathbb{D}$ be a packet zero (a candidate off-critical zeta zero coordinate). The *kernel evaluation functional* on the model space K_B is

$$\mathrm{ev}_\alpha \colon K_B \to \mathbb{C}, \qquad \mathrm{ev}_\alpha([f]) := \langle f,\, k_\alpha \rangle_{H^2},$$

where k_α is the reproducing kernel of the orthogonal complement of BH^2 at the point α.

Definition 2.16 (Rational dilation map)**.** Fix a (set-theoretic) function $Q \colon \mathbb{D} \to (0,1] \cap \mathbb{Q}$ associating each disk point with a representative rational dilation parameter. In practice we choose $Q(\alpha) := q$ where q is a fixed dyadic rational within the radius of α (concretely: $Q(\alpha) := 2^{-\lceil \log_2(1/|\alpha|) \rceil} \in (0,1] \cap \mathbb{Q}$). No continuity or smoothness of Q is required. This is recorded in the Lean development as the field `parameterOf`: `Detector` $\to \Theta$.

Theorem 2.17 (Mellin identification)**.** *Under the Mellin transform $\mathcal{M} \colon H^2 \to L^2((0,1])$, the kernel evaluation functional ev_α on K_B is proportional to the multiplicative pairing against the fractional-part atom $A_{Q(\alpha)}$:*

$$\langle f, k_\alpha \rangle_{H^2} = c_\alpha \int_0^1 (\mathcal{M}f)(x)\, \overline{A_{Q(\alpha)}(x)}\, \frac{dx}{x}$$

where $c_\alpha \in \mathbb{C}^\times$ is a normalizing constant depending only on the Mellin transform's normalization conventions (c_α may be taken equal to 1 when $\mathcal{M}$ is given the standard Burnol *normalization; we suppress its precise value because it has no bearing on the categorical content of the externalization). In particular, the family $\{A_{Q(\alpha)}\}_\alpha$ parameterises the RKHS detectors by rational dilation parameters.*

Sketch. This is the classical Burnol–Mellin transport: the inner product on H^2 is, after Mellin transform, the multiplicative inner product on $L^2((0,1])$, and the reproducing kernel k_α transports to (a constant multiple of) the fractional-part atom by the Burnol identity. We do not require the proof for the Lean formalization; it serves as the mathematical motivation for the *categorical* parametrization by rational dilation parameters. □

3 The Externalization Functor

3.1 The general RKHS detector externalization structure

We recall the central vol5 structure that this paper inhabits.

Definition 3.1 (`RKHSDetectorExternalizationToRationalRepresentables`)**.** For K a condensed Hilbert defect, S an RKHS model-space detector on K, and R a rational dilation object semantics (equipped with a parameter type $R.\texttt{Parameter}$ and an embedding $R.\mathsf{obj}_Q\colon R.\texttt{Parameter} \to D_Q$), an *externalization* of S to rational representables is a quadruple

(1) $\texttt{parameterOf}\colon S.\texttt{Detector} \to R.\texttt{Parameter}$,

(2) $\texttt{presheafOf}\colon S.\texttt{Detector} \to \mathbf{PSh}(D_Q)$,

(3) $\texttt{presheaf\_eq}\colon \forall d,\ \texttt{presheafOf}\ d = \mathrm{y}(R.\mathsf{obj}_Q(\texttt{parameterOf}\ d))$,

(4) $\texttt{detected\_of\_detector}\colon \forall v, d,\ S.\texttt{detectsVector}\ v\ d \to \texttt{CondensedHilbertDefectDetectedByRati}$

3.2 Worked example: a one-element packet

Example 3.2 (Externalization of a one-element packet)**.** Let $P = \{\alpha\}$ be a single-zero packet, with $\alpha \in \mathbb{D}$ representing a candidate off-critical zero. The finite Blaschke product is

$$B_\alpha(z) = \frac{z-\alpha}{1 - z\,\overline{\alpha}},$$

the model space $K_{B_\alpha} = H^2/B_\alpha H^2$ is one-dimensional, spanned by the kernel vector $k_\alpha^{K_B}$. The single-element detector is

$$S_\alpha = \{\mathrm{ev}_\alpha\colon K_{B_\alpha} \to \mathbb{C}\}, \qquad \mathrm{ev}_\alpha(c \cdot k_\alpha^{K_B}) = c \cdot \|k_\alpha^{K_B}\|^2,$$

which is non-zero on every non-zero vector by definition. The externalization sends ev_α to the parameter $\theta_\alpha := Q(\alpha) \in \Theta$ and the presheaf $\mathrm{y}(\mathsf{obj}_Q(\theta_\alpha)) \in \mathbf{PSh}(D_Q)$. The `detected_of_detector` field records:

$$\mathrm{ev}_\alpha(v) \neq 0 \implies \texttt{CondensedHilbertDefectDetectedByRationalDilation}\ K_B\ \mathrm{y}(\mathsf{obj}_Q(\theta_\alpha)),$$

which (in the conditional setting of theorem 3.4) is closed by the introduction rule ι.

3.3 Categorical structure: presheaves on D_Q

The full externalization picture is the diagram

$$\begin{tikzcd} S.\mathtt{Detector} \arrow[r, "\mathtt{parameterOf}"] \arrow[d, "\mathtt{presheafOf}"'] & \Theta \arrow[d, "R.\mathsf{obj}_Q"] \\ \mathbf{PSh}(D_Q) & D_Q \arrow[l, "\mathrm{y}"] \end{tikzcd}$$

required to commute up to the equation `presheaf_eq`. The data of `detected_of_detector` is then a natural compatibility: internal detection of v by index d implies external detection by the presheaf $\mathrm{y}(R.\mathsf{obj}_Q(\theta_d))$.

3.4 Conditional externalization theorem

We now state the principal theorem of this paper as a conditional externalization, parameterized on an introduction rule for the opaque predicate. In section 4 we discuss why the unconditional form is currently blocked.

Definition 3.3 (Externalization introduction rule)**.** An *externalization introduction rule* for a condensed Hilbert defect K (and an implicit rational dilation semantics R) is a function

$$\iota\colon \forall\, F \in \mathbf{PSh}(D_Q),\ \mathtt{Representable}\ F \to \mathtt{CondensedHilbertDefectDetectedByRationalDilation}\ K\ F,$$

where $\mathtt{Representable}\ F := \exists X \in D_Q,\ F = \mathrm{y}(X)$. The Lean formalization records ι as the field `intro` of the structure `ExternalizationIntroductionRule` K, taking `Representable` F as its input rather than the raw existential.

Theorem 3.4 (Conditional constant-presheaf externalization)**.** ***Caveat (theorem 3.7):*** *this theorem inhabits the externalization* type *via a* constant *construction (`parameterOf` and `presheafOf` are both constant on the detector index). It is not the analytic externalization that one would obtain from a non-trivial constructor for the opaque predicate `DefectDetectedByRationalDilationRepresentable`; rather, it gives the maximal type-theoretic inhabitant available under the externalization-introduction-rule data.*

Let K be a condensed Hilbert defect, S an RKHS model-space detector on K, and R a rational dilation object semantics with a designated parameter $\theta_0 \in R.$`Parameter`. Suppose furthermore:

(i) there exists an externalization introduction rule ι as in theorem 3.3.

Then there exists an inhabitant

$$E\colon \mathtt{RKHSDetectorExternalizationToRationalRepresentables}\ K\ S\ R$$

with

(1) `parameterOf` $d := \theta_0$ (constant in d),

(2) `presheafOf` $d := \mathrm{y}(R.\mathsf{obj}_Q(\theta_0))$ (constant in d),

(3) `presheaf_eq` *holds by reflexivity,*

(4) `detected_of_detector` *is given by ι applied to the (trivial) Yoneda witness for* $\mathrm{y}(R.\mathsf{obj}_Q(\theta_0))$.

Proof. We construct E field-by-field. For the parameter map, fix any $\theta_0 \in R.$`Parameter` supplied as data of R (the vol5 structure `RationalDilationObjectSemantics` provides the type $R.$`Parameter` but does not require it to be inhabited; we add the inhabitation as an explicit hypothesis if needed). Define

$$\texttt{parameterOf}\ d := \theta_0, \qquad \texttt{presheafOf}\ d := \mathrm{y}(R.\mathsf{obj}_Q(\theta_0)).$$

Then `presheaf_eq` d is the equation $\mathrm{y}(R.\mathsf{obj}_Q(\theta_0)) = \mathrm{y}(R.\mathsf{obj}_Q(\theta_0))$, which is rfl. Finally, given $v \in K.$`modelCarrier` and $d \in S.$`Detector` with $S.$`detectsVector` $v\ d$, we apply ι to the witness

$$\exists X \in D_Q.\ \mathrm{y}(R.\mathsf{obj}_Q(\theta_0)) = \mathrm{y}(X)$$

which is satisfied by $X := R.\mathsf{obj}_Q(\theta_0)$ and rfl. □

3.5 Trivial-detector externalization theorem

To produce a no-argument inhabitant of the externalization structure, we instantiate the conditional theorem at a detector whose `detectsVector` predicate is universally false.

Definition 3.5 (Empty detector)**.** For any condensed Hilbert defect K such that $K.$`modelCarrier` is empty, the *empty detector* is the RKHS model-space detector with

- `Detector` := `Empty`,
- `detectsVector` $v\ d$ – vacuous (no d exists),
- `vector_of_nonzero` – vacuous (no nonzero v exists),
- `detector_complete` v – vacuous (no v exists).

Theorem 3.6 (Trivial externalization)**.** *For any condensed Hilbert defect K with empty model carrier and any rational dilation object semantics R with non-empty parameter type, the empty detector externalizes:*

$$\textit{RKHSDetectorExternalizationToRationalRepresentables}\ K\ (\mathsf{empty})\ R.$$

Proof. The detector type is `Empty`, so all four fields of the externalization structure are vacuous: there is no detector index d to map and no detection event to externalize. Specifically:

- `parameterOf` : `Empty` $\to R.$`Parameter` – the function out of the empty type, `Empty.elim`.
- `presheafOf` : `Empty` $\to \mathbf{PSh}(D_Q)$ – again `Empty.elim`.
- `presheaf_eq` d – universal quantification over `Empty`, vacuous.
- `detected_of_detector` – universal quantification over $d \in$ `Empty`, vacuous.

The construction uses no axiom or opaque introduction rule. □

Remark 3.7 (Strength of the trivial theorem). Theorem 3.6 is the strongest unconditional externalization theorem currently constructible without modifying vol5. It shows that, as a categorical/type-theoretic statement, the externalization *structure* can be inhabited by a vacuous detector at the level of generality of theorem 3.1. The mathematical content of externalization (matching internal detection events to external presheaves) only becomes non-trivial once the detector is non-empty, which requires a nonempty model carrier and (more importantly) a constructor for the opaque predicate `DefectDetectedByRationalDilationRepresentable`. The obstruction to that constructor is the subject of section 4.

3.6 The principal theorem and its conditional form

Theorem 3.8 (Principal externalization theorem)**.** *Suppose we are given:*

(1) the canonical Burnol/Blaschke condensed Hilbert defect $K_B :=$ ***`burnolBlaschkeCondensedHilbertD`***

(2) an RKHS detector S on K_B (e.g. the canonical detector of Paper 02 of the sprint, when constructed),

*(3) a rational dilation object semantics R with a designated rational parameter $\theta_0 \in R.$**`Parameter`**,*

(4) an externalization introduction rule ι for K_B and R.

Then ***`RKHSDetectorExternalizationToRationalRepresentables`*** *K_B S R is inhabited.*

Proof. Apply theorem 3.4 with $K = K_B$ and the given R, S, and ι. □

4 The Externalization Obstruction

4.1 The opaque predicate

The vol5 module `NoPhantomBlaschkeDefect.lean` declares

`opaque DefectDetectedByRationalDilationRepresentable : BlaschkeDefectObject` $\to \mathbf{PSh}(D_Q) \to$

Opaque declarations in Lean 4 are propositions or types whose internal structure is sealed; only the declared type is visible. Consequently, no unfolding-based proof can produce an inhabitant of the predicate.

4.2 Survey of the vol5 surface

A direct `grep` of the vol5 source tree shows that the only references to `DefectDetectedByRationalDilatio`
are

- the declaration in `NoPhantomBlaschkeDefect.lean` line 28,
- negative occurrences in `FiniteRankRKHSDetector.lean` lines 51 and 98 (used as the *conclusion* of detector externalization or the *negation* of detection by orthogonal presheaves),

- occurrences in `NoPhantomBlaschkeDefect.lean` lines 38, 46, 110, 120 (similar negative or hypothesis-side uses),
- the unfolding in `CondensedHilbertDefect.lean` line 45.

There is no introduction rule, characterization theorem, or proof constructor for the opaque predicate exposed in the vol5 surface.

4.3 Characterization of the missing rule

Definition 4.1 (Externalization obstruction package)**.** The *externalization obstruction* for the canonical Burnol/Blaschke defect is the following package of data, which would suffice to close the externalization unconditionally:

(I) an introduction rule

$$\iota_{K_B} : \forall F \in \mathbf{PSh}(D_Q),\ \texttt{Representable}\ F \to \texttt{DefectDetectedByRationalDilationRepresentable b}$$

or,

(II) a characterization equation

$$\texttt{DefectDetectedByRationalDilationRepresentable}\ K\ F \longleftrightarrow \exists v \in K.\texttt{modelCarrier},\ \texttt{realize}\ F$$

relating the opaque predicate to a transparent inner-product statement, exposed as a theorem in vol5.

Remark 4.2. We emphasise that adding either (I) or (II) is forbidden by the Vol6 charter: it would amount to modifying vol5 (forbidden) or adding a new `axiom/opaque` (also forbidden). The obstruction package is therefore documentation of what *would* suffice if a future vol5 update made one of these forms transparent.

4.4 Comparison with the finite-rank detector externalization

The vol5 module `FiniteRankRKHSDetector.lean` contains an analogous structure `DetectorExternalizat` for *finite-rank* detectors (i.e. those with detector type Fin$r$ for some natural number r). This finite-rank version has the same four fields as theorem 3.1 but with the detector type restricted. The two structures are related by the embedding `rkhs_externalization_of_finite`, which lifts a finite-rank externalization to a general RKHS externalization.

In particular, any finite-rank externalization automatically inhabits the general structure when the detector index type Fin$r$ is non-empty (i.e. $r \geq 1$) and the introduction-rule data is supplied. This is the natural source of finite-rank RKHS externalization data once Paper 02 of the sprint constructs the canonical finite-rank Burnol/Blaschke detector.

4.5 The opaque predicate's logical role

The predicate `DefectDetectedByRationalDilationRepresentable` K F has a specific logical role in the No-Phantom Language: it is the hypothesis of the `NoPhantomBlaschkeDefectRigidity` principle, which asserts that any rational-dilation representable that detects an admissible defect must already lie in the Nyman/Yoneda representable image. The contradiction from a hypothetical off-critical zero arises by combining:

- the externalization theorem (this paper) producing a representable F that detects the defect,
- the rigidity principle forcing F to be Nyman-image,
- the quotient orthogonality (Paper 05) forbidding any Nyman-image presheaf from detecting the defect,

yielding a clash. For this contradiction to bite, the externalization must produce a *specific* representable F for each detector event, not a constant one. The constant form of theorem 3.4 is therefore not yet sufficient for the final RH derivation; it is a type-theoretic skeleton that becomes informative once a real introduction rule replaces the constant choice.

4.6 Why the trivial detector evades the obstruction

The obstruction module hands back the trivial-detector externalization theorem (theorem 3.6) as the maximal externalization inhabitant currently provable. It does so by:

- supplying the empty detector (theorem 3.5),
- supplying any rational dilation object semantics with a nonempty parameter type, e.g. R.`Parameter` := `Unit` and $R.\mathsf{obj}_Q\ u := \mathsf{base} \in D_Q$,
- observing that all four externalization fields collapse to vacuous quantifications.

This produces a sorry-free, axiom-free inhabitant of `RKHSDetectorExternalizationToRationalRepresent` which is all that the present module needs to deliver.

5 Lean Formalization

5.1 Module structure

The vol6 deliverable consists of two Lean modules:

- `Vol6/RationalDilationExternalization.lean` – contains the principal theorem `burnol_blaschke_` together with all auxiliary definitions and the trivial-detector instance;
- `Vol6/Obstruction/RationalDilationExternalizationObstruction.lean` – documents the externalization obstruction (theorem 4.1) and exposes the explicit `ExternalizationObstru` structure for downstream use.

5.2 Imports

The principal module imports:

```
import Vol5.YonedaNymanTrackA.RKHSDetectorSemantics
import Vol5.YonedaNymanTrackA.CondensedHilbertDefect
import Vol5.YonedaNymanTrackA.NoPhantomBlaschkeDefect
import Vol5.YonedaNymanTrackA.FiniteRankRKHSDetector
import Vol5.YonedaNymanTrackA.YonedaCore
import Vol5.YonedaNymanTrackA.FractionalPartExternalization
import Vol6.Obstruction.RationalDilationExternalizationObstruction
```

No vol6 paper 02 module is imported (it is not yet a deliverable in this paper 04 sprint).

5.3 Key definitions

In Lean syntax (specialized to the canonical Burnol/Blaschke defect):

```
/-- Empty detector for the canonical Burnol/Blaschke condensed
    Hilbert defect, valid under InternalBlaschkeTriviality. -/
noncomputable def burnolBlaschkeEmptyDetector
    (h : InternalBlaschkeTriviality) :
    RKHSModelSpaceDetector burnolBlaschkeCondensedHilbertDefect :=
  { Detector := Empty,
    detectsVector := fun _ d => Empty.elim d,
    vector_of_nonzero := fun hNonzero =>
      absurd
        (blaschke_defect_object_is_zero_of_internal_triviality h)
        hNonzero,
    detector_complete := fun v => (h.false v).elim }
```

The auxiliary lemma `blaschke_defect_object_is_zero_of_internal_triviality` is exposed by Vol5's `HardyModelSpace.lean` and converts `InternalBlaschkeTriviality` (i.e. emptiness of `burnolBlaschkeFactor.modelSpaceCarrier`) into `BlaschkeDefectObjectIsZero`, which definitionally unfolds to `CondensedHilbertDefectIsZero burnolBlaschkeCondensedHilbertDefe`
The `absurd` combinator then derives the desired `Nonempty` from the contradiction.

5.4 The principal theorem in Lean

```
theorem burnol_blaschke_detector_externalization
    {S : RKHSModelSpaceDetector burnolBlaschkeCondensedHilbertDefect}
    {R : RationalDilationObjectSemantics}
    (theta0 : R.Parameter)
    (iota : ExternalizationIntroductionRule
              burnolBlaschkeCondensedHilbertDefect R) :
    RKHSDetectorExternalizationToRationalRepresentables
```

```
    burnolBlaschkeCondensedHilbertDefect S R :=
  { parameterOf := fun _ => theta0,
    presheafOf := fun _ =>
      yoneda (R.objectOf theta0),
    presheaf_eq := fun _ => rfl,
    detected_of_detector := fun _ =>
      iota.intro (yoneda (R.objectOf theta0))
        (yoneda_is_representable _) }
```

5.5 The unconditional inhabitant

The unconditional version uses the empty detector for the canonical defect (under `InternalBlaschkeTrivia`

```
noncomputable def burnol_blaschke_trivial_externalization
    (h : InternalBlaschkeTriviality)
    (X : DQObj) :
    RKHSDetectorExternalizationToRationalRepresentables
      burnolBlaschkeCondensedHilbertDefect
      (burnolBlaschkeEmptyDetector h)
      (canonicalRationalDilationSemantics X) := ...
```

5.6 No-axiom, no-sorry, no-opaque verification

The Lean module passes the build target

```
PATH="$HOME/.elan/bin:$PATH" lake build Vol6.RationalDilationExternalization
```

with no `sorry`, no `admit`, no new `axiom`, and no new `opaque`. The propagated axioms (visible via `#print axioms`) are inherited from vol5 and Mathlib only.

6 Audit Trail

6.1 Forbidden imports

The principal Lean module `Vol6.RationalDilationExternalization` explicitly does *not* import:

- `Vol5.YonedaNymanTrackA.BurnolFactorization` (which contains the Nyman density / Burnol bridge equivalence), nor
- `Vol5.YonedaNymanTrackA.Criterion` (which contains the Beurling–Nyman criterion statement).

The transitive import set, computed by Lake's import graph, is restricted to vol5's D_Q-Yoneda surface, the condensed Hilbert defect API, and the RKHS detector semantics layer. No density-theoretic equivalent appears as a direct or indirect dependency.

6.2 Axioms inherited from vol5

Running `#print axioms` on the principal theorem `burnol_blaschke_detector_externalization` reveals only the following inherited axioms (all from vol5 and Mathlib):

- Mathlib's classical-logic core (`Classical.choice`, `propext`, `Quot.sound`).
- Vol5's D_Q-Yoneda axioms (`NymanKernel`, `DQHom`, `DQ_id`, `DQ_comp`).
- Vol5's Hardy-side axioms (`HardySpaceH2`, `HardySubmodule`, `blaschkeHardyImage`, `burnolBlaschkeFactor`).

None of `NymanDensity`, `NymanClosure`, `ClassicalNymanSpanMellinImage`, `BurnolFactorization` appears. This is the audit certification required by the Vol6 charter.

6.3 Status string

The Lean module exports an audit-readable status string

`rational_dilation_externalization_status`

together with a `rational_dilation_externalization_status_eq` theorem proving the string is the documented sentence. The obstruction module similarly exports `externalization_obstruction_sta` and its `_eq` theorem. These markers serve as machine-checkable audit anchors for downstream tooling.

7 Numerical Demonstration

The accompanying Haskell program `haskell/paper-04-rational-dilation-externalization/Main.hs` provides a concrete numerical demonstration of the fractional-part kernel calculation.

7.1 Algorithm

For a fixed rational $q \in (0,1) \cap \mathbb{Q}$ and indices $n = 1, \ldots, N$ with $N = 16$ (representing a finite Blaschke packet of size N), the program computes:

(1) the fractional-part values $\{q \cdot n\}$,

(2) the *discrete* Mellin pairing

$$M_q^{\mathrm{disc}}[k] := \frac{1}{N} \sum_{n=1}^{N} \{qn\} \cdot k(n/N)$$

against a finite-packet kernel $k\colon (0,1] \to \mathbb{R}$. This is a Riemann-sum approximation (with respect to ordinary Lebesgue measure dx) of an integral pairing of the fractional-part sequence with the kernel; it is *not* the analytic Mellin pairing of theorem 2.17 (whose multiplicative measure is dx/x). We use it here purely for numerical illustration of the parametrization $q \mapsto \{\{qn\}\}_{n=1}^{N}$;

(3) the *refined-grid reference* value

$$M_q^{\mathrm{ref}}[k] := \frac{1}{NM}\sum_{n=1}^{NM}\{qn\}\cdot k(n/(NM))$$

with refinement factor $M = 64$, i.e. the same Riemann sum on a $64\times$ finer grid.

The agreement

$$\left|M_q^{\mathrm{disc}}[k] - M_q^{\mathrm{ref}}[k]\right| < \epsilon$$

(for $\epsilon = 0.05$ on a 16-point packet) confirms numerical stability of the parametrization $q \mapsto \{qn\}$ at this resolution.

7.2 Implementation in Haskell

The Haskell program is structured as a library plus a small driver:

- `fractionalPartKernel q n` returns the list $[\{q\cdot 1\}, \{q\cdot 2\}, \ldots, \{q\cdot n\}]$.
- `mellinPairingDiscrete q n k` computes $M_q^{\mathrm{disc}}[k]$ at resolution n.
- `analyticMellinReference q n k` computes $M_q^{\mathrm{ref}}[k]$ at $64\times$ finer resolution.
- `runTest` compares the two and reports whether the difference falls below the tolerance.

The driver runs five test cases (constant, linear, quadratic kernels across $q \in \{1/3, 1/2, 2/5, 1/4\}$) and reports a summary `passed/total` count.

7.3 Sample output

For $q = 1/3$ and the kernel $k(x) = 1$ (constant), the cycle $\{n/3\}$ over $n = 1, \ldots, 16$ is $1/3, 2/3, 0, 1/3, 2/3, 0, \ldots, 1/3$ (5 complete cycles plus one extra), so

$$M_{1/3}[1] = \frac{1}{16}\sum_{n=1}^{16}\{n/3\} = \frac{1}{16}\cdot\frac{16}{3} = \frac{1}{3}.$$

For $q = 1/2$ and $k(x) = 1$, the cycle $\{n/2\}$ over $n = 1, \ldots, 16$ is $1/2, 0, 1/2, 0, \ldots, 1/2, 0$ (8 complete pairs of 1/2 and 0), so

$$M_{1/2}[1] = \frac{1}{16}\sum_{n=1}^{16}\{n/2\} = \frac{1}{16}\cdot 4 = \frac{1}{4} = 0.25.$$

These match the analytic mean of $\{qn\}$ over a complete period.

8 Results

8.1 Formal results

Theorem 8.1 (Conditional externalization, formal)**.** *The Lean module `Vol6.RationalDilationExternal` contains a sorry-free proof of the conditional externalization theorem `burnol_blaschke_detector_externa` parameterized by an externalization introduction rule.*

Theorem 8.2 (Trivial externalization, formal)**.** *The Lean module `Vol6.RationalDilationExternalizat` contains a sorry-free proof of the unconditional trivial externalization `burnol_blaschke_trivial_externa` valid for any condensed Hilbert defect with empty model carrier.*

Theorem 8.3 (Obstruction characterization, formal)**.** *The Lean module `Vol6.Obstruction.RationalDil` exposes the `ExternalizationObstructionPackage` structure together with a proof that any inhabitant of the package supplies the externalization introduction rule.*

8.2 Numerical results

Theorem 8.4 (Numerical Mellin pairing agreement)**.** *The Haskell program `haskell/paper-04-rational-` when run with $q = 1/3$ and the constant kernel, computes a Mellin pairing agreeing with the analytic reference to within tolerance $\epsilon = 0.05$ on a 16-point packet.*

9 Discussion

9.1 Position in the Vol6 sprint

Paper 04 sits between Paper 02 (finite-rank RKHS detector construction) and Paper 05 (quotient orthogonality and admissibility) in the vol6 sprint sequence. Its role is the categorical bridge: once Paper 02 produces a finite-rank detector $S \in$ `FiniteModelSpaceDetector` (K_B), Paper 04 externalizes S to a family of D_Q-Yoneda representables. The output feeds into Paper 06 (assembly of `YonedaBlaschkeDetectorCalculus`).

9.2 What this paper does and does not prove

Proved.

- The categorical/type-theoretic skeleton of externalization: every internal detector index has an externalized presheaf which is a Yoneda representable.
- The conditional externalization theorem (theorem 3.4), parameterized by an introduction rule.
- The unconditional trivial externalization theorem (theorem 3.6), valid for empty model carriers.
- The obstruction characterization (theorem 8.3).

Not proved (intentionally deferred to vol5 surface extension).

- Concrete inhabitants of `DefectDetectedByRationalDilationRepresentable` K_B F for non-trivial detector presheaves F. This is blocked by the opaque declaration in vol5 (section 4).

9.3 Path forward

The cleanest path to closing the externalization unconditionally is a vol5 patch exposing the introduction rule of theorem 4.1 (I) or the characterization equation (II). Either form keeps the vol5 hard rules (no new `axiom`, no new `opaque`, no `sorry`, no `admit`) intact while unlocking the present paper's theorem in non-conditional form.

9.4 Relationship to classical Beurling–Nyman

The fractional-part atoms A_θ used in the Mellin picture are the same atoms that appear in the classical Beurling–Nyman criterion for RH. However, our use of them is different: we use only the *parametrization* $\theta \mapsto A_\theta$, never the *density* of their span in $L^2((0,1])$. In particular, no result of the present paper depends on the Nyman–Beurling density theorem or any of its equivalents. This is essential for the Vol6 hard rules.

10 Conclusion

We have:

- formulated the externalization functor as a categorical bridge from internal RKHS detectors to external D_Q-presheaves;
- identified the rational dilation parameter $Q(\alpha)$ for each detector index, via the Mellin/multiplicative coordinate;
- proved a conditional externalization theorem, sorry-free, parameterized by an introduction rule for the opaque `DefectDetectedByRationalDilationRepresentable`;
- produced an unconditional trivial-detector inhabitant of the externalization structure;
- characterized the precise obstruction to closing the externalization unconditionally.

The Lean modules `Vol6.RationalDilationExternalization` and `Vol6.Obstruction.RationalDilatio` build cleanly under the Vol6 hard rules. The Haskell module `haskell/paper-04-rational-dilation-exte` demonstrates the Mellin pairing numerically.

References

References

[1] J.-F. Burnol, *The Riemann zeta function and the orthogonal polynomial method*. Manuscript, 2010.

[2] B. Nyman, *On the One-Dimensional Translation Group and Semi-Group in Certain Function Spaces*. PhD thesis, Uppsala University, 1950.

[3] A. Beurling, *A closure problem related to the Riemann zeta-function*, Proc. Nat. Acad. Sci. USA **41** (1955), 312–314.

[4] J. Baez and J. Dolan, *From finite sets to Feynman diagrams*. In: Mathematics Unlimited - 2001 and Beyond, Springer, 2001.

[5] S. Mac Lane, *Categories for the Working Mathematician*, 2nd edition. Springer, 1998.

[6] Vol5 source: `hott-riemann-hypothesis-vol5/lean/Vol5/YonedaNymanTrackA/`.

[7] Vol5 `sources/next-steps.md`, §2.2 *Rational-Dilation Externalization* and Recommended Proof Sprint Order item 4.

[8] Vol5 `.knowledge-base.md` (Vol6 mirror).

[9] The Lean 4 Theorem Prover team, *Lean 4 Reference Manual*, 2024.

[10] The mathlib Community, *The Lean Mathematical Library*. https://leanprover-community.github.io/